CHAPTER 8. STOCHASTIC SIMULATION WITH AN APPLICATION TO A RBC MODEL

1. Theory

In chapter 7 we introduced the perfect foresight macroeconomic model:

(1) $F(y_t, y_{t+1}, y_{t-1}, z_t) = 0$, y_0 given, $t \geq 1$.

F is a vector of n equations, y is the column vector of the n endogenous variables, z is the column vector of the m exogenous variables.

The time-path followed by the vector of exogenous variables is arbitrary (under the constraint that it is bounded). The agents in the model, and the economists, who use it perfectly know this time-path (or at least assume that they know it perfectly). So, there is no uncertainty. This kind of model is useful to investigate the effects of structural reforms, as changing the tax system or trade policy. It can also be used to build scenarios, such that the effects of an increase in the price of oil.

In this chapter we introduce the model:

(2) $E_t\left\{F(y_t, y_{t+1}, y_{t-1}, u_t)\right\} = 0$,

u_t is a vector of exogenous stochastic shocks, of dimension m, which is independently and identically distributed, with its two first moments: $Eu_t = 0$ and $Eu_t u_t' = \Sigma$. E is the operator of unconditional expected value. E_t is the operator of expected value conditional to the information available at time t, which are the current and past values of y and u. So, the agents in the model, and the macroeconomists who use the model, do not know the future values of the shocks. We express the difference between the two models by calling the former, deterministic, and the latter, stochastic.

We look for the stationary stochastic process, y_t, which satisfies the model. More precisely, we can show that this stochastic process that is the solution of the model can be represented by the policy function:

(3) $y_t = g(y_{t-1}, u_t)$.

When we have computed this function, we can easily run stochastic simulations of it (after having drawn a pseudo random sequence of u_t. If we have run a large number of stochastic simulations, for instance 500, we can use them to estimate the moments of the stochastic process y_t, especially its covariance and autocovariance matrices.

This kind of model is useful to investigate the cyclical properties of the economy, and to identify policy rules, which limit the amplitude of the cycles followed by the main variables (production, price, employment). These rules may specify the monetary policy (as the Taylor rule) or the fiscal policy (as the rules in the Maastricht treaty).

The steady state equilibrium of the model, $u_t = \bar{u}$ (no shock) and $y_t = \bar{y}$ (the endogenous variables do not change over time) is such that

(4) $F(\bar{y}, \bar{y}, \bar{y}, 0) = 0$, and $\bar{y} = g(\bar{y}, 0)$.

Let $\overline{F_1}$, $\overline{F_2}$, $\overline{F_3}$ and $\overline{F_4}$ be the four partial derivatives matrices of $F(y_t, y_{t+1}, y_{t-1}, u_t)$, computed at the steady state. The dimensions of the three first matrices is $n*n$. The dimension of the last matrix is $n*m$. $\bar{g}_1$ and $\bar{g}_2$ are the two partial derivatives matrices of $g(y_{t-1}, u_t)$, computed at the steady state. Their respective dimensions are $n*n$ and $n*m$.

We deduce from equation (3):

(5) $y_{t+1} = g(y_t, u_{t+1}) = g[g(y_{t-1}, u_t), u_{t+1}]$.

We substitute equations (3) and (5) in equation (2):

(6) $E_t\{F\{g(y_{t-1}, u_t), g[g(y_{t-1}, u_t), u_{t+1}], y_{t-1}, u_t\}\} = 0$.

We define $\hat{y}_{t-1} = y_{t-1} - \bar{y}$ and we remind that $F(\bar{y}, \bar{y}, \bar{y}, 0) = 0$. The first-order (linear) approximation of this equation in the neighbourhood of the steady state is:

(7) $E_t\left\{\overline{F}_1\left(\overline{g}_1\hat{y}_{t-1}+\overline{g}_2u_t\right)+\overline{F}_2\left[\overline{g}_1\left(\overline{g}_1\hat{y}_{t-1}+\overline{g}_2u_t\right)+\overline{g}_2u_{t+1}\right]+\overline{F}_3\hat{y}_{t-1}+\overline{F}_4u_t\right\}=0$.

At time t, the value of $\hat{y}_{t-1}$ and u_t are known, but not the value of u_{t+1}. So, we have: $E_t\left(\hat{y}_{t-1}\right)=\hat{y}_{t-1}$, $E_t\left(u_t\right)=u_t$ and $E_t\left(u_{t+1}\right)=0$. Of course, the values of $\overline{F}_1$, $\overline{F}_2$, $\overline{F}_3$ and $\overline{F}_4$, and of $\overline{g}_1$ and $\overline{g}_2$, are also known. We deduce from equation (7):

(8) $\overline{F}_1\left(\overline{g}_1\hat{y}_{t-1}+\overline{g}_2u_t\right)+\overline{F}_2\overline{g}_1\left(\overline{g}_1\hat{y}_{t-1}+\overline{g}_2u_t\right)+\overline{F}_3\hat{y}_{t-1}+\overline{F}_4u_t=0$,

or

(9) $\left(\overline{F}_1\overline{g}_1+\overline{F}_2\overline{g}_1\overline{g}_1+\overline{F}_3\right)\hat{y}_{t-1}+\left(\overline{F}_1\overline{g}_2+\overline{F}_2\overline{g}_1\overline{g}_2+\overline{F}_4\right)u_t=0$.

The linear approximation of the policy function (3) is:

(10) $\hat{y}_t=\overline{g}_1\hat{y}_{t-1}+\hat{g}_2u_t$

$\overline{g}_1$ is a $n*n$ matrix and $\overline{g}_2$ is a $n*m$ matrix. Equation (9) must be satisfied for any value of $\hat{y}_{t-1}$ and u_t. This implies that their factor terms must be zero:

(11a) $\overline{F}_1\overline{g}_1+\overline{F}_2\overline{g}_1\overline{g}_1+\overline{F}_3=0$

(11b) $\overline{F}_1\overline{g}_2+\overline{F}_2\overline{g}_1\overline{g}_2+\overline{F}_4=0$

We compute $\overline{g}_1$ from equation (10a) and then, we easily deduce the value of $\overline{g}_2$ from equation (10b). Equation (10a) yields a quadratic expression in $\overline{g}_1$, which can be solved with a series of algebraic tricks (such as the Schur generalized decomposition). Incidentally, one of the conditions that come out of the solution of this equation is the Blanchard and Kahn condition: there must be as many roots larger than one in modulus as there are forward-looking variables in the model.

If we are interested in impulse response functions, we simply iterate the policy function starting from an initial value given by the steady state.

The covariance matrix of the shocks, especially their individual variances, plays no role in the first order approximation. That means that risk plays no role, which prevents us to investigate questions such as precautionary saving, or equities risk premiums, or the welfare gain induced by the stabilisation of the economy. For this reason, macroeconomists prefer to use a second order approximation of functions F and g. The notations and the algebra become more cumbersome with a second order approximation, but the principles of the computation are the same.

2. Solving a real business cycle model

Households maximize utility over consumption, c_t and leisure, $1-l_t$, where l_t is labor input, according to the following utility function:

$$\max E \sum_{t=0}^{\infty} \beta \left\{ \log c_t + \psi \log \left(1 - l_t\right) \right\}$$

and subject to the following budget constraint:

$$c_t + k_{t+1} = w_t l_t + r_t k_t + \left(1 - \delta\right) k_t, \; \forall \; t > 0$$

where k_t is capital stock, w_t real wages, r_t real interest rates or cost of capital and δ the depreciation rate. This equation can be seen as an accounting identity, with total expenditures on the left hand side and revenues - including the liquidation value of the capital stock - on the right hand side.

Maximization of the household problem with respect to consumption, leisure and capital stock, yields the Euler equation in consumption, capturing the intertemporal tradeoff between consumption and investment, and the labour supply equation linking labour positively to wages and negatively to consumption (the wealthier, the more leisure due to the decreasing marginal utility of consumption). These equations are:

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \left(1 + r_{t+1} - \delta \right) \right]$$, and :

$$\psi \frac{c_t}{1 - l_t} = w_t$$

The firm side of the problem is slightly more involved, due to monopolistic competition. We postulate that there is a continuum of intermediate producers with market power who each sell a different variety to a competitive final goods producer whose production function is a CD aggregate of intermediate varieties. The final goods producer chooses his or her optimal demand for each variety, yielding the Dixit-Stiglitz downward sloping demand curve. Intermediate producers, instead, face a two pronged decision: how much labour and capital to employ given these factors' perfectly competitive prices and how to price the variety they produce.

Production of intermediate goods follows a CD production function defined as:

$$y_{it} = k_{it}^{\alpha} \left(e^{z_t} l_{it} \right)^{1-\alpha}$$

where the *i* subscript stands for firm *i* of a continuum of firms between zero and one and where α is the capital elasticity in the production function, with $0 < \alpha < 1$. Also, z_t captures technology which evolves according to

$$z_t = \rho z_{t-1} + \varepsilon_t$$

where ρ is a parameter capturing the persistence of technological progress and

$$\varepsilon_t \sim \mathcal{N}(0, \sigma)$$

The solution to the sourcing problem yields an optimal capital to labour ratio, or relationship between payments to factors:

$$k_{it} r_t = \frac{\alpha}{1-\alpha} w_t l_{it}$$

The solution to the pricing problem, instead, yields the well-known constant markup pricing condition of monopolistic competition:

$$p_{it} = \frac{\varepsilon}{1-\varepsilon} mc_t p_t$$

where p_{it} is firm i's specic price, mc_t is real marginal cost and p_t is the aggregate price or average price. An additional step simplifies this expression: symmetric firms implies that all firms charge the same price and thus $p_{it} = p_t$; we therefore have:

$$mc_t = (\varepsilon - 1)/\varepsilon \text{ .}$$

What are marginal costs equal to? To find the answer, we combine the optimal capital to labour ratio into the production function and solve for the amount of labour or capital required to produce one unit of output. The real cost of using this amount of any one factor is given by $w_t l_{it} + r_t k_{it}$ where we substitute out the payments to the other factor using again the optimal capital to labour ratio. When solving for labour, for instance, we obtain

$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{1}{A_t} w_t^{1-\alpha} r_t^{\alpha}$$

which does not depend on *i*; it is thus the same for all firms.

Combining this result for marginal cost, as well as its counterpart in terms of capital, with the optimal pricing condition yields the final two important equations of our model

$$w_t = (1-\alpha)\frac{y_{it}}{l_{it}}\frac{\varepsilon - 1}{\varepsilon}$$

and

$$r_t = \alpha \frac{y_{it}}{k_{it}} \frac{\varepsilon - 1}{\varepsilon}$$

To end, we aggregate the production of each individual firm to find an aggregate production function. On the supply side, we factor out the capital to labor ratio, k_t / l_t, which is the same for all firms and thus does not depend on *i*. On the other side, we have the Dixit-Stiglitz demand for each variety. By equating the two and integrating both side, and noting that $p_{it} = p_t$ we obtain aggregate production

$$y_t = k_t^{\alpha} \left(e^{z_t} l_t\right)^{1-\alpha}$$

which can be shown is equal to the aggregate amount of varieties bought by the final good producer (according to the aggregation index) and, in turn, equal to the aggregate output of final good, itself equal to household consumption. Note that because the ratio of output to each factor is the same for each intermediate firm, we can rewrite the above two equations for w_t and r_t without the *i* subscripts on the right hand side.

The program of this model is written in rbc_monopolistic.mod

References

Griffoli, Tommaso Mancini. (2007-2008). *Dynare user guide. An introducyion to the solution & estimation of DSGE models*. www.dynare.org.

Fernandez-Villaverde. *RBC models*. www.dynare.org.