CHAPTER 7. THE RAMSEY'S MODEL AND

HOW TO SIMULATE A PERFECT FORESIGHT DYNAMIC MODEL

In this chapter we will use the freeware DYNARE, which works under MATLAB. DYNARE has a website www.dynare.org. I use version 7.0 of MATLAB and the version 4.21 of DYNARE.
DYNARE has met a lot of success with central banks, which have developed dynamic stochastic general equilibrium models inspired by the works of Christiano and Eichembaum in the US and Smets and Wouters in Europe. DYNARE is very useful to simulate these models, but also to estimate their parameters by using Bayesian econometrics. I will also use a paper of mine: "A Sufficient Condition for the Existence and the Uniqueness of a Solution in Macroeconomic Models with Perfect Foresight", *Journal of Dynamic Economics and Control*, 28, p. 1955-1975, 2004.

You have to unzip DYNARE in a specific directory and add a path to this directory in MATLAB. You will find a tutorial and plenty of materials in the zip file.

We will start by an example to motivate the theoretical developments, which will follow.

1. An example: the Ramsey's model

This model is the most famous and simplest dynamic general equilibrium model. We will successively give the list of the equations, the endogenous variables, the exogenous variables and the parameters. Then, we will explain the economic meaning of the model.

Equations

(1) $c_t + k_t - (1-\delta)k_{t-1} = ax_t k_{t-1}^{\alpha}$

(2) $Max \sum_{t=0}^{\infty} (1+\beta)^{-t} c_t^{1-\gamma} / (1-\gamma)$, with $t \geq 0$ and k_{-1} given

Endogenous variables

c_t : Households' consumption in period t

k_t : Productive capital at the end of period t

Exogenous variable

x_t : Productivity shock in period t

Parameters

$0 < \alpha < 1$: Share of labour income in total production

$0 < \delta < 1$: Capital depreciation rate

$\gamma > 0$: Relative risk aversion

$\beta > 0$: Households' discount rate

a : Coefficient of the production function

Comments

The model represents a closed economy. The first equation is the equilibrium of the good market. The various components of demand appear on the left-hand side. Supply is given on the right-hand side. We assume full employment and the supply of labour is normed to 1. Under the assumptions of the model, especially the absence of distortions, the market equilibrium is identical to the optimum. This one is given by the maximisation of the welfare function of the representative household, which is given in the second expression of the model

First order condition (FOC)

There is one equation (1) in each period. Let us call λ_t the Lagrange multiplier of this equation at period *t*. We write the Lagrangian of the maximisation program. Then, we compute its derivatives relative to c_t, then to k_t and put them equal to 0:

(3) $(1+\beta)^{-t} c_t^{-\gamma} + \lambda = 0$

(4) $\lambda_t - \left[(1-\delta) + a x_{t+1} \alpha k_t^{\alpha-1}\right] \lambda_{t+1} = 0$

Finally, we use equation (3) written in periods *t* and *t+1*, to eliminate λ_t and λ_{t+1} from equation (4). We get:

(5) $c_t^{-\gamma} - \left[(1-\delta) + a x_{t+1} \alpha k_t^{\alpha-1}\right](1+\beta)^{-1} c_{t+1}^{-\gamma} = 0$

Simulation of the Ramsey's model: transitory shock

There will be no confusion if we consider that period *t* is the current period and we put no index for the variables in this period. Variables at time *t-1* will be indexed by -1. Variables at time *t+1* will be indexed by +1. So, the model, that is equations (1) and (5), can be written:

(1) $c + k - (1-\delta)k_{-1} = a x k_{-1}^{\alpha}$

(5) $c^{-\gamma} - \left[(1-\delta) + a x_{+1} \alpha k^{\alpha-1}\right](1+\beta)^{-1} c_{+1}^{-\gamma} = 0$

We will assume that at time 0, we have: $x_0 = \bar{x} = 1$, and that the economy is in its steady state defined by:

(6) $\bar{c} + \bar{k} - (1-\delta)\bar{k}_{-1} = a \bar{x} \bar{k}_{-1}^{\alpha}$

(7) $\bar{c}^{-\gamma} - \left[(1-\delta) + a \bar{x} \alpha \bar{k}^{\alpha-1}\right](1+\beta)^{-1} \bar{c}^{-\gamma} = 0$

Then, in period 1 we will set: $x_1 = 1.2$ In the following periods, we will set again variable *x* to its former value: $x_t = \bar{x} = 1$. So, the economy is shocked by a productivity shock of amplitude 20%, but for only one period. We want to simulate the paths of capital and consumption from time 1 to infinity.

This problem is non classical. The current economic situation, that is the values of c and k, depends on the past through the value of k_{-1}, but also on the (expected) future, through the value of c_{+1}. So, we cannot proceed as with old fashioned models where variables of the current period and of past periods, but not of future periods, appear. For these models we can proceed recursively. First, we solve the model at period 1 that is we compute the values of the variables in this period. Then we solve the model in period 2 that is we compute the values of the model in this period. Etc. Here we must solve all the periods simultaneously, and cannot proceed recursively.

Simulation of the Ramsey's model: permanent shock

This time, in period 1 *and afterward* we will set: $x_{:1} = 1.2$. So, the economy is still shocked by a productivity shock of amplitude 20%, but for all periods until infinity. It will converge to a new steady state. We want to simulate the paths of capital and consumption from time 1 to infinity.

2. Linear dynamic models with perfect foresight

A dynamic linear model with perfect foresight can be specified in three forms. The first called the *structural form*, is the direct expression of the economics features of the problem and has a clear and simple economic interpretation. DYNARE is user-friendly enough to deal with this form.

However, this form is unpractical for the mathematical analysis of the model. A *canonical form* can be deduced from the structural form. Then, we can deduce from this form the *Blanchard and Kahn's reduced form*, which is extremely useful to determine if the model has a solution and if this solution is unique.

The general ***structural form*** of a discrete time linear model with perfect foresight is:

$$(1)\quad \sum_{k=0}^{K}\sum_{h=0}^{H} B_{kh\ t-k} Z_{t+h-k} = L_t$$

${}_{t-k}Z_{t+h-k}$ represents the forecast of the vector of the endogenous variables at time $(t+h-k)$ conditionally to the information available at time $(t-k)$. B_{kh} is a matrix of fixed parameters, and L_t is a vector of exogenous variables. The forecast of a variable for the current period or for past periods is equal to the true value of this variable at this time: ${}_{t-k}Z_{t-k-l} = Z_{t-k-l}$ for

$l \geq 0$. Thus, the model written at time t includes the values taken by some variables at time t or before and the forecast of variables made at time t or before for later periods.

A necessary condition for the model to give a consistent and complete description of the economy, is that it determines without ambiguity the current state of the economy when the past values and the past and current forecasts of the endogenous variables, and the values of the exogenous variables, are given. So, we impose the condition that matrix B_{00} is non singular *(at a practical level, this condition is very important; beginners who fail when they use DYNARE, mostly fail because they use a lousy model; so the first thing they have to do is finding what is wrong with their model).*

We can show that the addition of artificial variables allows rewriting the general structural form in a *canonical form*

(2) $C_1 y_{t-1}^1 + C_0 y_t + C_{-1}\, {}_t y_{t+1}^2 = U_t$, with C_0 non singular

The new endogenous variables which appear in this form belong to one of three mutually exclusive classes. m_1 variables appear in a contemporary or lagged form; They are denoted as *predetermined*. m_2 other variables appear in a contemporary or led form; They are denoted as *anticipated*. The m_3 last variables only appear in a contemporary form; They are denoted as *static*. These three categories of variables constitute at time t the column vectors y_t^1, y_t^2 and y_t^3. ${}_t y_{t+1}^2$ is the forecast of y_{t+1}^2 at time t. The piling up of these vectors in the order: y_t^2, y_t^3 and y_t^1 defines the vector of the endogenous variables y_t of dimension: $m = m_1 + m_2 + m_3$.

We are interested by the forecasts made at time 0 with equation (2). These forecasts satisfy:

(3) $C_1\, {}_0 y_{t-1}^1 + C_0\, {}_0 y_t + C_{-1}\, {}_0 y_{t+1}^2 = {}_0 U_t$, for $t \geq 1$.

In the rest of the chapter we will omit the pre-index 0 from the notations. We shall assume that the exogenous variables, U_t are constant over time and will denote them by U.

To get the *Blanchard and Kahn's reduced form*, let us denote by: C_0^2, C_0^3 and C_0^1 the three matrices with respectively m_2, m_3 and m_1 columns., such that their concatenation gives the matrix $C_0 : C_0 = (C_0^2 \,|\, C_0^3 \,|\, C_0^1)$. Let us make the change of variables: $x_t^1 = y_t^1$, $x_t^2 = y_{t+1}^2$, $x_t^3 = y_t^3$, and let us denote the piling up of these vectors in the order: x_t^3, x_t^1, x_t^2, by x_t. Then, the model can be rewritten:

(4) $C_1 x_{t-1}^1 + C_0^2 x_{t-1}^2 + (C_0^3 \,|\, C_0^1 \,|\, C_{-1}) x_t = U$.

In general, for x_{t-1}^1 and x_{t-1}^2 given, equation (4) does not determine a unique value for x_t. However, it is possible to make a series of eliminations and transformations of anticipated variables and put the model in the case where the uniqueness of x_t is warranted. DYNARE includes a solution to this problem. It rests upon the computation of *generalised eigenvalues* based on a generalised decomposition of Schur. Then, the variables with a lead which can be eliminated are as many as there are infinite eigenvalues. We can also eliminate all the static variables of the model. Finally we get the Blanchard and Kahn's reduced form:

(5) $x_t = Ax_{t-1} + h$, $t \geq 1$.

x_t denotes the vector of the endogenous variables, h is a vector and A a non singular square matrix. All these are of dimension n. h and A are constant over time.
For the linear approximation of the Ramsey model in the neighbourhood of its steady state, equation (5) is

$$\begin{pmatrix} c_{t+1} \\ k_t \end{pmatrix} = A \begin{pmatrix} c_t \\ k_{t-1} \end{pmatrix} + h$$

We saw that a part of the endogenous variables are predetermined, and the others are anticipated. We make the assumption that the values of the predetermined variables at time 0 are known. However, we cannot assume that the initial values of the anticipated variables are known. This ignorance may potentially lead to an indeterminate solution. To solve this problem, economists added conditions that the paths followed by the endogenous variables must satisfy. The *asymptotic stability of this path* will be the condition that we introduce here.

We will assume that the first n_1 components of x_t are predetermined variables the initial values of which x_0^1 are given. The last $n-n_1$ components of x_t are not predetermined and their initial values are free to jump at the initial time.

The steady state model associated with equation (5): $(I-A)\bar{x}=h$, determines the steady states of the economy $\bar{x}$. We will assume that this steady state exists and is unique, which means that matrix *I-A* is regular or matrix *A* has no eigenvalue equal to one. We have the definition

Definition 1. An asymptotically stable solution to the model in equation (5) is a path of vector: x_t, $t \geq 0$, *which satisfies equation (5), such that* x_0^1 *is given, and such that* x_t *will tend to satisfy the steady state model when time approaches infinity:* $(I-A)x_t \to h$, *when* $t \to \infty$

Under technical assumptions that I will not give here we can prove the following proposition

Proposition 1. The necessary and sufficient condition for the existence and uniqueness of an asymptotically stable solution to the model in equation (5), requires that the number of eigenvalues of matrix A with absolute values less than 1 must equal the number of predetermined variables.

3. Nonlinear models with perfect foresight

A perfect foresight model can be written as[1]:

(6) $F(y_t, y_{t+1}, y_{t-1}, z_t) = 0$, y_0 given, $t \geq 1$.

F is a vector of n equations, y is the column vector of the n endogenous variables, z is the column vector of the m exogenous variables. The endogenous variables belong to one of the three mutually exclusive classes: either they appear in a contemporary or lagged form and are denoted as *predetermined;* or they appear in a contemporary or lead form and are denoted as *anticipated*: or they only appear in a contemporary form and are denoted as *static*. At time t,

[1] If the structural model has lags and leads longer than 1, we can add new artificial variables, as in section 2, to put the model in the form of equation 6.

the model determines the current values of the endogenous variables y_t in function of the values of these variables which are anticipated for the future or which were observed in the past, y_{t+1} and y_{t-1}, and of the values of the exogenous variables z_t.

We set $z_t = \bar{z}$ and we assume that for every vector of exogenous variables $\bar{z}$, the model in equation (6) has a steady state $(\bar{z}, \bar{y})$ determined by the steady state model: $F(\bar{y}, \bar{y}, \bar{y}, \bar{z}) = 0$. We can extend Definition 1 in the following way

Definition 2. An asymptotically stable solution of model in equation (6) is a path of the endogenous vector: y_t, $t \geq 1$, *which satisfies equation (6), for a given* y_0, *and such that* y_t *will tend to satisfy the steady state model when time approaches infinity:*

(7) $F(y_t, y_t, y_t, \bar{z}) \rightarrow 0$, *when* $t \rightarrow \infty$.

Such a solution can numerically be computed with DYNARE provided a unique solution exists. As the model is non-linear, to be able to use the results of existence and uniqueness of section 2, we will look for solutions in the neighbourhood of a steady state, and substitute the model by a linear approximation. Let us introduce the square matrices of dimension n of the partial derivatives of function F:

$$F_i^{'}(y_1, y_2, y_3, z) = \frac{\partial F(y_1, y_2, y_3, z)}{\partial y_i}, \; i = 1,2,3.$$

Then, we can compute the linear approximation of the model in the neighbourhood of a steady state

(8) $F_1^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z})(y_t - \bar{y}) + F_2^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z})(y_{t+1} - \bar{y}) + F_3^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z})(y_{t-1} - \bar{y}) = 0$

The linear approximation of the stability condition (7) is:

(9) $[F_1^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z}) + F_2^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z})g + F_3^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z})](y_t - \bar{y}) \equiv B(y_t - \bar{y}) \rightarrow 0$, when $t \rightarrow \infty$

Then we can apply Proposition 1 to this case and, under some technical assumptions that will not be developed here, we get

Proposition 2. If the linear approximation of the model has as many eigenvalues of absolute values less than 1 as there exists predetermined variables, then the linear approximation of the model has a unique asymptotically stable solution.

4. How to simulate a non linear model with perfect foresight

We have to simulate the model

(10) $F(y_t, y^1_{t+1}, y^2_{t-1}, z_t, a) = 0$, y_0 given, $t \geq 1$

y_t is a vector of n endogenous variables, which is split between the three sub-vectors y^1_t, y^2_t and y^3_t. The variables in y^1_t are the anticipated variables, which appear with indices t and t+1. The variables in y^2_t are the predetermined variables, which appear with indices t and t-1. The variables in y^3_t only appear with index t and are the static variables.

To simulate the model we must add initial conditions, for instance the value of the predetermined variables in period 0: y^2_0. In general we assume that the model was in period 0 in the steady state associated to an initial value $\underline{x}$ of the exogenous variable and defined by $F\left(\underline{y}^1, \underline{y}, \underline{y}^2, \underline{x}, a\right) = 0$. Thus, we solve this system of n equations, with unknown variables $\underline{y}$, and we set $y^2_0 = \underline{y}^2$.

We must also add terminal conditions, for instance the assumption that the anticipated variables will not diverge to infinity when time increases indefinitely.

In general we simulate the model for $t = 1,..,T$, where T is a large number (100 periods or more), which approximates infinity. We assume that the exogenous variables have been set to a final value $\bar{x}$ after period T_1, with $T_1 << T$. Then we compute the final steady state values of the endogenous variables by solving the system $F\left(\bar{y}^1, \bar{y}, \bar{y}^2, \bar{x}, a\right) = 0$ and we set $y^1_{T+1} = \bar{y}^1$.

The best method to simulate the model is to stack all the equations of all periods. Then, we have a system of $n(T+1)$ equations with $n(T+1)$ variables, y_t for $0 \leq t \leq T$. This system looks huge: if we have a model with 20 equations and if we simulate it over 100 periods, we have to solve a system of 2000 equations with 2000 variables. However, few variables appear in a given equation. For instance in an equation of period t, only *some* variables of periods *t-1*, *t* and *t+1* will appear. DYNARE uses this feature to simplify the computations. Otherwise, DYNARE uses a Newton's method, with a *LU* decomposition in each iteration (see chapters 2 and 3). Thus, the above simulation takes no more than a few seconds.

Before simulating the model, we must set the values of the parameters and of the exogenous variables. We want that the equations, which define its steady state, can reproduce the average situation of the economy. This average situation of the economy is defined by the values of the endogenous and exogenous variables, which are observable (e.g. GDP is observable, but not the total productivity of factors). So we want the equations of the model to be satisfied when some endogenous and exogenous variables haves been set at their observed values

(11) $F(y,x,a)=0$

Moreover, the values of some parameters are known (for instance they were computed by an econometric estimation).

There are n equations in system (11), and $n+m+p$ exogenous variables, endogenous variables and parameters. So, we can set the values of $m+p$ of these variables and parameters, and use system (11) to compute the values of the other variables and parameters.

Thus, to calibrate a model we have to solve a system of n equations with n variables, which is mathematically the same thing as simulating a static model.

At the end of the calibration stage we have defined a reference state of the economy. If the model is dynamic, the reference state will (most often) be used as initial condition for the predetermined variables of the model.

5. Application to the Ramsey's model

We will use DYNARE, which works under MATLAB. We have to write our problem in a specific file with the attribute.mod, in conformity with a specific syntax. Then at the command line of MATLAB we will type DYNARE followed by the name of the file.

The steady state model

We remind that the equations of the model are

(1) $c_t + k_t - (1-\delta)k_{t-1} = ax_t k_{t-1}^{\alpha}$

(5) $c_t^{-\gamma} - \left[(1-\delta) + ax_{t+1}\alpha k_t^{\alpha-1}\right](1+\beta)^{-1} c_{t+1}^{-\gamma} = 0$

To write the steady state model we remove all time indices

(1) $c + k - (1-\delta)k = axk^{\alpha}$

(5) $c^{-\gamma} - \left[(1-\delta) + ax\alpha k^{\alpha-1}\right](1+\beta)^{-1} c^{-\gamma} = 0$

Equation (5) gives $1+\beta-(1-\delta) = ax\alpha k^{\alpha-1}$

and determine sthe steady state value of capital. Then equation (1) determines the steady state value of consumption.

The code of the model

```
//We write the name of the endogenous variables, of the exogenous variables and of the parameters
var c k;
varexo x;
parameters alph gam delt bet a;

//We write the numerical values of the parameters
alph=0.5;
gam=0.5;
delt=0.02;
bet=0.05;
a=0.5;

//We write the equations of the model
model;
c + k - (1-delt)*k(-1)=a*x*k(-1)^alph ;
c^(-gam) = (1+bet)^(-1)*(a*alph*x(+1)*k^(alph-1) + 1 - delt)*c(+1)^(-gam);
```

```
end;

//We write the initial values - that is the values at time -1 - of the
//exogenous and endogenous variables.
//The only initial value, which really matters is the initial value of the capital stock
//The others will be used as initial guesses when we simulate the model
//We have assumed that the initial state of the economy is a steady state
initval;
x = 1;
k = ((delt+bet)/(1.0*a*alph))^(1/(alph-1));
c = a*k^alph-delt*k;
end;
//If we had been too lazy to compute the solution of the steady state
//MATLAB would have computed it for us with the following command
steady;
//The next command computes the eigenvalues of the model
//We will see that they are important to determine if the model has a unique solution
check;

//We introduce the one period productivity shock
shocks;
var x;
periods 1;
values 1.2;
end;

//We introduce the permanent productivity shock
//endval;
//x = 1.2;
//k = ((delt+bet)/(1.0*a*alph))^(1/(alph-1));
//c = a*k^alph-delt*k;
//end;
//We compute the new steady sate
//the previous values of k and g are used as initial guesses in this computation
```

```
//Here the guesses are right, but this is nor the case in general
//steady;

//We run the simulation
simul(periods=200);

//We get the graphs of the results
rplot c;
rplot k;
```

The results of the simulation

```
>> Dynare ramsey
Warning: Function call Dynare invokes inexact match
D:\Dynare_v3.065\dynare_v3\matlab\dynare.m.



STEADY-STATE RESULTS:

c          1.53061
k          12.7551

EIGENVALUES:
     Modulus      Real     Imaginary


      0.935      0.935          0
      1.123      1.123          0



There are 1 eigenvalue(s) larger than 1 in modulus
for 1 forward-looking variable(s)

The rank condition is verified.

----------------------------------------------------
```

MODEL SIMULATION :

1 - err = 0.31802
Time of iteration :0.32
2 - err = 0.00052099
Time of iteration :0.115
3 - err = 1.3103e-008
Time of iteration :0.091

Total time of simulation :0.674

Convergency obtained.
