CHAPTER 6. COMPARATIVE DYNAMICS OR

HOW TO SIMULATE A PERFECT FORESIGHT DYNAMIC MODEL

In this chapter we will use the freeware DYNARE, which works under MATLAB. DYNARE has a website http://www.cepremap.cnrs.fr/dynare/. I emailed to you version 3_065 because I had problems with version 4, which might have a bug. DYNARE has met a lot of success with central banks, which have developed dynamic stochastic general equilibrium models inspired by the works of Christiano and Eichembaum in the US and Smets and Wouters in Europe. DYNARE is very useful to simulate these models, but also to estimate their parameters by using Bayesian econometrics. I will also use a paper of mine: "A Sufficient Condition for the Existence and the Uniqueness of a Solution in Macroeconomic Models with Perfect Foresight", *Journal of Dynamic Economics and Control*, 28, p. 1955-1975, 2004.

You have to unzip DYNARE in a specific directory and add a path to this directory in MATLAB. You will find a tutorial and plenty of materials in the zip file.

We will start by an example to motivate the theoretical developments, which will follow.

1. An example: the Ramsey's model

This model is the most famous and simplest dynamic general equilibrium model. As in chapter 4 we will successively give the list of the equations, the endogenous variables, the exogenous variables and the parameters. Then, we will explain the economic meaning of the model.

Equations

(1) $c_t + k_t - (1-\delta)k_{t-1} = a x_t k_{t-1}^{\alpha}$

(2) $Max \sum_{t=0}^{\infty} (1+\beta)^{-t} c_t^{1-\gamma} / (1-\gamma)$, with $t \geq 0$ and k_{-1} given

Endogenous variables

c_t : Households' consumption in period t

k_t : Productive capital at the end of period t

Exogenous variable

x_t : Productivity shock in period t

Parameters

$0 < \alpha < 1$: Share of labour income in total production

$0 < \delta < 1$: Capital depreciation rate

$\gamma > 0$: Relative risk aversion

$\beta > 0$: Households' discount rate

a : Coefficient of the production function

Comments

The model represents a closed economy. The first equation is the equilibrium of the good market. The various components of demand appear on the left-hand side. Supply is given on the right-hand side. We assume full employment and the supply of labour is normed to 1. Under the assumptions of the model, especially the absence of distortions, the market equilibrium is identical to the optimum. This one is given by the maximisation of the welfare function of the representative household, which is given in the second expression of the model

First order condition (FOC)

There is one equation (1) in each period. Let us call λ_t the Lagrange multiplier of this equation at period *t*. We write the Lagrangian of the maximisation program. Then, we compute its derivatives relative to c_t, then to k_t and put them equal to 0:

(3) $(1+\beta)^{-t} c_t^{-\gamma} + \lambda = 0$

(4) $\lambda_t - \left[(1-\delta) + a x_{t+1} \alpha k_t^{\alpha-1}\right] \lambda_{t+1} = 0$

Finally, we use equation (3) written in periods *t* and *t+1*, to eliminate λ_t and λ_{t+1} from equation (4). We get:

(5) $c_t^{-\gamma} - \left[(1-\delta) + ax_{t+1}\alpha k_t^{\alpha-1}\right](1+\beta)^{-1} c_{t+1}^{-\gamma} = 0$

Simulation of the Ramsey's model: transitory shock

There will be no confusion if we consider that period *t* is the current period and we put no index for the variables in this period. Variables at time *t-1* will be indexed by -1. Variables at time *t+1* will be indexed by +1. So, the model, that is equations (1) and (5), can be written:

(1) $c + k - (1-\delta)k_{-1} = axk_{-1}^{\alpha}$
(5) $c^{-\gamma} - \left[(1-\delta) + ax_{+1}\alpha k^{\alpha-1}\right](1+\beta)^{-1} c_{+1}^{-\gamma} = 0$

We will assume that at time 0, we have: $x_0 = \bar{x} = 1$, and that the economy is in its steady state defined by:

(6) $\bar{c} + \bar{k} - (1-\delta)\bar{k}_{-1} = a\bar{x}\bar{k}_{-1}^{\alpha}$
(7) $\bar{c}^{-\gamma} - \left[(1-\delta) + a\bar{x}\alpha\bar{k}^{\alpha-1}\right](1+\beta)^{-1}\bar{c}^{-\gamma} = 0$

Then, in period 1 we will set: $x_1 = 1.2$ In the following periods, we will set again variable *x* to its former value: $x_t = \bar{x} = 1$. So, the economy is shocked by a productivity shock of amplitude 20%, but for only one period. We want to simulate the paths of capital and consumption from time 1 to infinity.

This problem is non classical. The current economic situation, that is the values of c and k, depend on the past through the value of k_{-1}, but also on the (expected) future, through the value of c_{+1}. So, we cannot proceed as with old fashioned models where variables of the current period and of past periods, but not of future periods, appear. For these models we can proceed recursively. First, we solve the model at period 1 that is we compute the values of the variables in this period. Then we solve the model in period 2 that is we compute the values of the model in this period. Etc. Here we must solve all the periods simultaneously, and cannot proceed recursively.

We will look at the MATLAB program ramsey.mod, which I put in the working directory.

Simulation of the Ramsey's model: permanent shock

This time, in period 1 *and afterward* we will set: $x_{:1} = 1.2$. So, the economy is still shocked by a productivity shock of amplitude 20%, but for all periods until infinity. It will converge to a new steady state. We want to simulate the paths of capital and consumption from time 1 to infinity.

2. Linear dynamic models with perfect foresight. Theory

A dynamic linear model with perfect foresight can be specified in three forms. The first called the *structural form*, is the direct expression of the economics features of the problem and has a clear and simple economic interpretation. DYNARE is user-friendly enough to deal with this form.

However, this form is unpractical for the mathematical analysis of the model. A *canonical f*orm can be deduced from the structural form. Then, we can deduce from this form the *Blanchard and Kahn's reduced form*, which is extremely useful to determine if the model has a solution and if this solution is unique.

The general ***structural form*** of a discrete time linear model with perfect foresight is:

(1) $\sum_{k=0}^{K}\sum_{h=0}^{H} B_{kh\ t-k} Z_{t+h-k} = L_t$

${}_{t-k}Z_{t+h-k}$ represents the forecast of the vector of the endogenous variables at time $(t+h-k)$ conditionally to the information available at time $(t-k)$. B_{kh} is a matrix of fixed parameters, and L_t is a vector of exogenous variables. The forecast of a variable for the current period or for past periods is equal to the true value of this variable at this time: ${}_{t-k}Z_{t-k-l} = Z_{t-k-l}$ for

$l \geq 0$. Thus, the model written at time t includes the values taken by some variables at time t or before and the forecast of variables made at time t or before for later periods.

A necessary condition for the model to give a consistent and complete description of the economy, is that it determines without ambiguity the current state of the economy when the past values and the past and current forecasts of the endogenous variables, and the values of the exogenous variables, are given. So, we will impose the condition that matrix B_{00} is non singular *(at a practical level, this condition is very important; beginners who fail when they use DYNARE, mostly fail because they use a lousy model; so the first thing they have to do is finding what is wrong with their model).*

We can show that the addition of artificial variables allows rewriting the general structural form in a *canonical form*

(2) $C_1 y^1_{t-1} + C_0 y_t + C_{-1}\,{}_t y^2_{t+1} = U_t$, with C_0 non singular

The new endogenous variables which appear in this form belong to one of three mutually exclusive classes. m_1 variables appear in a contemporary or lagged form; They are denoted as *predetermined*. m_2 other variables appear in a contemporary or led form; They are denoted as *anticipated*. The m_3 last variables only appear in a contemporary form; They are denoted as *static*. These three categories of variables constitute at time t the column vectors y^1_t, y^2_t and y^3_t. ${}_t y^2_{t+1}$ is the forecast of y^2_{t+1} at time t. The piling up of these vectors in the order: y^2_t, y^3_t and y^1_t defines the vector of the endogenous variables y_t of dimension: $m = m_1 + m_2 + m_3$.

We are interested by the forecasts made at time 0 with equation (2). These forecasts satisfy:

(3) $C_{1\,0} y^1_{t-1} + C_{0\,0} y_t + C_{-1\,0} y^2_{t+1} = {}_0 U_t$, for $t \geq 1$.

In the rest of the chapter we will omit the pre-index 0 from the notations. We shall assume that the exogenous variables, U_t are constant over time and will denote them by U.

To get the *Blanchard and Kahn's reduced form*, let us denote by: C_0^2, C_0^3 and C_0^1 the three matrices with respectively m_2, m_3 and m_1 columns., such that their concatenation gives the matrix $C_0 : C_0 = (C_0^2 \mid C_0^3 \mid C_0^1)$. Let us make the change of variables: $x_t^1 = y_t^1$, $x_t^2 = y_{t+1}^2$, $x_t^3 = y_t^3$, and let us denote the piling up of these vectors in the order: x_t^3, x_t^1, x_t^2, by x_t. Then, the model can be rewritten:

(4) $C_1 x_{t-1}^1 + C_0^2 x_{t-1}^2 + (C_0^3 \mid C_0^1 \mid C_{-1}) x_t = U$.

In general, for x_{t-1}^1 and x_{t-1}^2 given, equation (4) does not determine a unique value for x_t. However, it is possible to make a series of eliminations and transformations of anticipated variables and put the model in the case where the uniqueness of x_t is warranted. DYNARE includes a solution to this problem. It rests upon the computation of *generalised eigenvalues* based on a generalised decomposition of Schur. Then, the variables with a lead which can be eliminated are as many as there are infinite eigenvalues. We can also eliminate all the static variables of the model. Finally we get the Blanchard and Kahn's reduced form:

(5) $x_t = A x_{t-1} + h$, $t \geq 1$.

x_t denotes the vector of the endogenous variables, h is a vector and A a non singular square matrix. All these are of dimension n. h and A are constant over time.

We saw that a part of the endogenous variables are predetermined, and the others are anticipated. We make the assumption that the values of the predetermined variables at time 0 are known. However, we cannot assume that the initial values of the anticipated variables are known. This ignorance may potentially lead to an indeterminate solution. To solve this problem, economists added conditions that the paths followed by the endogenous variables must satisfy. The *asymptotic stability of this path* will be the condition that we introduce here.

We will assume that the first n_1 components of x_t are predetermined variables the initial values of which x_0^1 are given. The last $n - n_1$ components of x_t are not predetermined and their initial values are free to jump at the initial time.

The steady state model associated with equation (5): $(I-A)\bar{x}=h$, determines the steady states of the economy $\bar{x}$. We will assume that this steady state exists and is unique, which means that matrix *I-A* is regular or matrix *A* has no eigenvalue equal to one. We have the definition

Definition 1. An asymptotically stable solution to the model in equation (5) is a path of vector: x_t, $t \geq 0$, *which satisfies equation (5), such that* x_0^1 *is given, and such that* x_t *will tend to satisfy the steady state model when time approaches infinity:* $(I-A)x_t \rightarrow h$, *when* $t \rightarrow \infty$

Under technical assumptions that I will not give here we can prove the following proposition

Proposition 1. The necessary and sufficient condition for the existence and uniqueness of an asymptotically stable solution to the model in equation (5), requires that the number of eigenvalues of matrix A with absolute values less than 1 must equal the number of predetermined variables.

3. Nonlinear models with perfect foresight. Theory

A perfect foresight model can be written as[1]:

(6) $F(y_t, y_{t+1}, y_{t-1}, z_t)=0$, y_0 given, $t \geq 1$.

F is a vector of n equations, y is the column vector of the n endogenous variables, z is the column vector of the m exogenous variables. The endogenous variables belong to one of the three mutually exclusive classes: either they appear in a contemporary or lagged form and are denoted as *predetermined;* or they appear in a contemporary or lead form and are denoted as *anticipated*: or they only appear in a contemporary form and are denoted as *static*. At time t, the model determines the current values of the endogenous variables y_t in function of the values of these variables which are anticipated for the future or which were observed in the past, y_{t+1} and y_{t-1}, and of the values of the exogenous variables z_t.

[1] If the structural model has lags and leads longer than 1, we can add new artificial variables, as in section 2, to put the model in the form of equation 6.

We set $z_t = \bar{z}$ and we assume that for every vector of exogenous variables $\bar{z}$, the model in equation (6) has a steady state $(\bar{z}, \bar{y})$ determined by the steady state model: $F(\bar{y}, \bar{y}, \bar{y}, \bar{z}) = 0$. We can extend Definition 1 in the following way

Definition 2. An asymptotically stable solution of model in equation (6) is a path of the endogenous vector: y_t, $t \geq 1$*, which satisfies equation (6), for a given* y_0*, and such that* y_t *will tend to satisfy the steady state model when time approaches infinity:*

(7) $F(y_t, y_t, y_t, \bar{z}) \to 0$*, when* $t \to \infty$.

Such a solution can numerically be computed with DYNARE provided a unique solution exists. As the model is non-linear, to be able to use the results of existence and uniqueness of section 2, we will look for solutions in the neighbourhood of a steady state, and substitute the model by a linear approximation. Let us introduce the square matrices of dimension n of the partial derivatives of function F :

$$F_i^{'}(y_1, y_2, y_3, z) = \frac{\partial F(y_1, y_2, y_3, z)}{\partial y_i},\ i = 1,2,3.$$

Then, we can compute the linear approximation of the model in the neighbourhood of a steady state

(8) $F_1^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z})(y_t - \bar{y}) + F_2^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z})(y_{t+1} - \bar{y}) F_3^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z})(y_{t-1} - \bar{y}) = 0$

The linear approximation of the stability condition (7) is:

(9) $[F_1^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z}) + F_2^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z}) g + F_3^{'}(\bar{y}, \bar{y}, \bar{y}, \bar{z})](y_t - \bar{y}) \equiv B(y_t - \bar{y}) \to 0$, when $t \to \infty$

Then we can apply Proposition 1 to this case and, under some technical assumptions that will not be developed here, we get

Proposition 2. If the linear approximation of the model has as many eigenvalues of absolute values less than 1 as there exists predetermined variables, then the linear approximation of the model has a unique asymptotically stable solution.

4. How to simulate a non linear model with perfect foresight

We have to simulate the model

(10) $F(y_t, y^1_{t+1}, y^2_{t-1}, z_t.a) = 0$, y_0 given, $t \geq 1$

y_t is a vector of n endogenous variables, which is split between the three sub-vectors y^1_t, y^2_t and y^3_t. The variables in y^1_t are the anticipated variables, which appear with indices *t* and *t*+1. The variables in y^2_t are the predetermined variables, which appear with indices *t* and *t*-1. The variables in y^3_t only appear with index *t* and are the static variables.

To simulate the model we must add initial conditions, for instance the value of the predetermined variables in period 0: y^2_0. In general we assume that the model was in period 0 in the steady state associated to an initial value $\underline{x}$ of the exogenous variable and defined by $F\left(\underline{y}^1, \underline{y}, \underline{y}^2, \underline{x}, a\right) = 0$. Thus, we solve this system of *n* equations, with unknown variables $\underline{y}$, and we set $y^2_{-1} = \underline{y}^2$.

We must also add terminal conditions, for instance the assumption that the anticipated variables will not diverge to infinity when time increases indefinitely.

In general we simulate the model for $t = 1,..,T$, where T is a large number (100 periods or more), which approximates infinity. We assume that the exogenous variables have been set to a final value $\bar{x}$ after period T_1, with $T_1 << T$. Then we compute the final steady state values of the endogenous variables by solving the system $F\left(\bar{y}^1, \bar{y}, \bar{y}^2, \bar{x}, a\right) = 0$ and we set $y^1_{T+1} = \bar{y}^1$.

The best method to simulate the model is to stack all the equations of all periods. Then, we have a system of $n(T+1)$ equations with $n(T+1)$ variables, y_t for $0 \leq t \leq T$. This system looks huge: if we have a model with 20 equations and if we simulate it over 100 periods, we have to solve a system of 2000 equations with 2000 variables. However, few variables appear in a given equation. For instance in an equation of period t, only *some* variables of periods *t-1*, *t* and *t+1* will appear. DYNARE uses this feature to simplify the computations. Otherwise, DYNARE uses a Newton's method, with a *LU* decomposition in each iteration (see chapters 2 and 3). Thus, the above simulation takes no more than a few seconds.

Before simulating the model, we must set the values of the parameters and of the exogenous variables. If the model is dynamic we want the equations, which define its steady state, can reproduce this average situation of the economy. This average situation of the economy is defined by the values of the endogenous and exogenous variables, which are observable (e.g. GDP is observable, but not the total productivity of factors). So we want the equations of the model to be satisfied when some endogenous and exogenous variables haves been set at their observed values

(11) $F(y,x,a)=0$

Moreover, the values of some parameters are known (for instance they were computed by an econometric estimation).

There are n equations in system (11), and $n+m+p$ exogenous variables, endogenous variables and parameters. So, we can set the values of $m+p$ of these variables and parameters, and use system (11) to compute the values of the other variables and parameters.

Thus, to calibrate a model we have to solve a system of n equations with n variables, which is mathematically the same thing as simulating a static model.

At the end of the calibration stage we have defined a reference state of the economy. If the model is dynamic, the reference state will (most often) be used as initial condition for the predetermined variables of the model.

5. An example

We will use DYNARE, which works under MATLAB. We have to write our problem in a specific file with the attribute.mod, in conformity with a specific syntax. Then at the command line of MATLAB we will type DYNARE followed by the name of the file.

The example is taken from "A Dynamic Analysis of Tied Aid", by Chi-Chur Chao, Bharat R. Hazari, Jean-Pierre Laffargue and Eden S. H. Yu, *Theory and Practice of Foreign Aid,* Sajal Lahiri ed., Elsevier, Amsterdam, 2007, 173-183.

The equations (15)

We have two sectors, agriculture and manufacturing, and two factors, labour and capital, which are perfectly mobile.

Short term equilibrium in the production sector (equations 1 to 8)

Firms act under perfect competition and benefit from an externality created by the total amount of capital available in the economy. This total capital $K_{-1} = K_X + K_Y$ is inherited from the past and is given. Capital can be freely allocated between both sectors. This is the reason why total capital is indexed by -1 (it is predetermined in the short term equilibrium) and the capital in each sector is not indexed (it is an endogenous variable in the short run equilibrium). Firms assume that the price of the manufactured good p and the cost of labour w impose to them. The production sector has to solve the program:

$$\underset{X,Y,L_X,L_Y,K_X K_Y}{Max} X + pY - w(L_X + L_Y)$$

$$X = A_X L_X^{\alpha_1} K_X^{\alpha_2}$$

$$Y = A_Y K_{-1}^{\beta_3} L_Y{}^{\beta_1} K_Y{}^{\beta_2}$$

$$K_{-1} = K_X + K_Y$$

We get the first order conditions

$$\alpha_1 A_X (K_X / L_X)^{\alpha_2} L_X^{\alpha_1+\alpha_2-1} = \beta_1 p A_Y K_{-1}^{\beta_3} \left(K_Y / L_Y\right)^{\beta_2} L_Y^{\beta_1+\beta_2-1} = w$$

$$\alpha_2 A_X K_{-1}^{\alpha_3} (L_X / K_X)^{\alpha_1} K_X^{\alpha_1+\alpha_2-1} = \beta_2 p A_Y K_{-1}^{\beta_3} \left(L_Y / K_Y\right)^{\beta_1} K_Y^{\beta_1+\beta_2-1} = r$$

By simple elimination we get the two price factor frontiers

$$\left(\frac{w}{\alpha_1}\right)^{1/\alpha_2} \left(\frac{r}{\alpha_2}\right)^{1/(1-\alpha_2)} L_X^{(1-\alpha_1-\alpha_2)/[\alpha_2(1-\alpha_2)]} = A_X{}^{1/[\alpha_2(1-\alpha_2)]}$$

$$\left(\frac{w}{\beta_1}\right)^{1/\beta_2} \left(\frac{r}{\beta_2}\right)^{1/(1-\beta_2)} L_Y^{(1-\beta_1-\beta_2)/[\beta_2(1-\beta_2)]} = \left(A_Y K_{-1}^{\beta_3}\right)^{1/[\beta_2(1-\beta_2)]} p$$

or

$$\left(\frac{w}{\alpha_1}\right)^{1-\alpha_2}\left(\frac{r}{\alpha_2}\right)^{\alpha_2} L_X^{(1-\alpha_1-\alpha_2)} = A_X$$

$$\left(\frac{w}{\beta_1}\right)^{1-\beta_2}\left(\frac{r}{\beta_2}\right)^{\beta_2} L_Y^{(1-\beta_1-\beta_2)} = A_Y K_{-1}^{\beta_3} p$$

Finally, we can write the eight first equations of the model

(1) $X = A_X L_X^{\alpha_1} K_X^{\alpha_2}$

(2) $Y = A_Y K_{-1}^{\beta_3} L_Y^{\beta_1} K_Y^{\beta_2}$

(3) $\alpha_1 A_X (K_X / L_X)^{\alpha_2} L_X^{\alpha_1+\alpha_2-1} = \beta_1 p A_Y K_{-1}^{\beta_3} (K_Y / L_Y)^{\beta_2} L_Y^{\beta_1+\beta_2-1}$

(4) $w = \alpha_1 A_X (K_X / L_X)^{\alpha_2} L_X^{\alpha_1+\alpha_2-1}$

(5) $K_{-1} = K_X + K_Y$

(6) $\left(\frac{w}{\alpha_1}\right)^{1-\alpha_2}\left(\frac{r}{\alpha_2}\right)^{\alpha_2} L_X^{(1-\alpha_1-\alpha_2)} = A_X$

(7) $\left(\frac{w}{\beta_1}\right)^{1-\beta_2}\left(\frac{r}{\beta_2}\right)^{\beta_2} L_Y^{1-\beta_1-\beta_2} = A_Y K_{-1}^{\beta_3} p$

(8) $L = L_X + L_Y$

Households' short run equilibrium (equations 9 and 10)

Households' consumption C aggregates exportable and importable goods $C = \left[b^{1/(1+\sigma)} C_X^{\sigma/(1+\sigma)} + \bar{b}^{1/(1+\sigma)} C_Y^{\sigma/(1+\sigma)}\right]^{(1+1/\sigma)}$, with $1+\sigma \geq 0$. σ is the elasticity of substitution between the two goods. $b \in [0,1]$ and $\bar{b} = 1-b$ are parameters. The composition of aggregate C results from the minimisation of its cost $C_Y + pC_Y$. We get the first order condition of the maximisation program $C = \frac{C_X}{b}\left(b + \bar{b} p^{-\sigma}\right)^{1+1/\sigma}$.

The current utility of households is: $U(C) = C^{(1-\lambda)}/(1-\lambda)$. $\lambda \geq 0$ is the inverse of the intertemporal substitution rate. We get $U(C) = \left[\frac{C_X}{b}\left(b + \bar{b} p^{-\sigma}\right)^{1+1/\sigma}\right]^{1-\lambda} /(1-\lambda)$.

Finally, we get two more equations of the model

(9) $\dfrac{bC_Y}{\overline{b}C_X} = \dfrac{1}{p^{1+\sigma}}$

(10) $U = \left[b^{1/(1+\sigma)}C_X^{\sigma/(1+\sigma)} + \overline{b}^{1/(1+\sigma)}C_Y^{\sigma/(1+\sigma)}\right]^{(1+1/\sigma)(1-\lambda)} /(1-\lambda)$

Equilibrium of the goods markets (equations 11 and 12)

The two equilibrium equations are.

(11) $K - K_{-!} + C_Y = Y + \overline{Q} + \beta T$

(12) $X + pY + (p - p^*)(\overline{Q} + \beta T) + p^*T = C_X + p(C_Y + K - K_{-1})$

There is an import quota on manufactured good $\overline{Q}$. T is foreign aid and β is the share of foreign aid, which must be spent on the imports of manufacturing good.

We can deduce from equations (11) and (12) the equation $p^*(\overline{Q} + \beta T) - (X - C_X) = p^*T$. The left-hand side is the commercial deficit of the country. It is the difference between imports and exports. The right-hand side is foreign aid. All variables are measured at the international price of the importable good (in terms of exportable good), and not at the price paid by the private sector.

Intertemporal households' decisions (equation 13)

The intertemporal utility of consumers is $\sum_{t=0}^{\infty}(1-\rho)^t\left[C_X\left(b + \overline{b}p^{-\sigma}\right)^{1+1/\sigma}\right]^{1-\gamma} /(1-\lambda)$. This function is maximised relatively to capital and the consumption of exportable good, in each period under the series of budget constraints:

$$p(K - K_{-1}) + C_X\left(b + \overline{b}p^{-\sigma}\right)/b = rK_{-1} + (p - p^*)(\overline{Q} + \beta T) + p^*T + wL$$

The rent generated by the quota and proportional to the difference between domestic and foreign price of the importable good, and foreign aid, are assumed to be distributed to households as lump sum transfers. The incomes of specific factors are also distributed as lump sum transfers and were omitted from the budget of households, because they do not interfere with their maximisation program. Let us solve the maximisation program. We get: $(1-\rho)^t C_X^{-\lambda}(b + \overline{b}p^{-\sigma})^{(1+1/\sigma)(1-\lambda)-1} = \mu / b$ (derivation relative to C_X), $\mu - \mu_{+1}\left(1 + r_{+1}\right) = 0$ (derivation relative to K). After the elimination of μ and μ_{+1} we get equation (13).

Finally, we get the equation

(13) $1+r_{+1}=\dfrac{1}{1-\rho}\left(\dfrac{C_X}{C_{X,+1}}\right)^{-\lambda}\dfrac{(b+\bar{b}p^{-\sigma})^{(1+1/\sigma)(1-\lambda)-1}}{(b+\bar{b}p_{+1}{}^{-\sigma})^{(1+1/\sigma)(1-\lambda)-1}}$

Equations(14) and (15)

The last equations control if the domestic price of the importable manufactured good is larger than the world price that is if the quota binds and if the net exports of the agricultural good are positive, along the dynamic path.

(14) $pr=p-p^*$

(15) $EX=X-C_X-(K-K_{-1})$

Endogenous variables (15)

X : production of exportable agricultural good

L_X : employment in the exportable agricultural good sector

Y : production of importable manufactured good

L_Y : employment in the importable manufactured good sector

K : total fixed capital *at the end of the period*

K_X : fixed capital in the exportable agricultural good sector *at the beginning* of the period

K_Y : fixed capital in the importable manufactured good sector *at the beginning* of the period

p : domestic price of the importable manufactured good (the exportable agricultural good is used as the numeraire)

w : cost of labour (the exportable good is used as the numeraire)

r : domestic interest rate (the exportable good is used as the numeraire)

C_Y : households' consumption of importable manufactured good

C_X : households' consumption of exportable agricultural good

U : current utility of households

m : direct impact of aid on the demand for the importable good

pr : premium on imports price

EX : exports of agricultural good

Exogenous variables

L : total employment = supply of labour

$\overline{Q} > 0$: the import quota on manufactured good, which is not connected to foreign aid

β : βT is the increase in the import quota resulting from foreign aid

$p*$: world price of the importable good (the exportable good is used as the numeraire)

T : foreign aid (the *importable* good is used as the numeraire)

Parameters

$0 < \alpha_1, \alpha_2, \beta_1, \beta_2$, $\alpha_1 + \alpha_2, \beta_1 + \beta_2 < 1$: shares of labour and capital income in the production of the both sectors

$\beta_2 > \alpha_2$: the importable good sector is more capital intensive than the exportable good sector

$\beta_3 \geq 0$, externality from total capital on the global productivity of factors in the manufactured goods sector

$A_X > 0$: global productivity of the exportable agricultural good sector

$A_Y > 0$: global productivity of the importable manufactured good sector

$0 < \rho < 1$: discount rate of households

$\lambda \geq 0$: inverse of the intertemporal rate of substitution of households

$1 + \sigma \geq 0$: elasticity of substitution between the two goods consumed by households.

$b \in [0,1]$ and $\overline{b} = 1 - b$, parameters in the households' utility function.

The steady state model

(1) $X = A_X L_X^{\alpha_1} K_X^{\alpha_2}$

(2) $Y = A_Y K^{\beta_3} L_Y{}^{\beta_1} K_Y{}^{\beta_2}$

(3) $\alpha_1 A_X (K_X / L_X)^{\alpha_2} L_X^{\alpha_1+\alpha_2-1} = \beta_1 p A_Y K^{\beta_3} \left(K_Y / L_Y\right)^{\beta_2} L_Y^{\beta_1+\beta_2-1}$

(4) $w = \alpha_1 A_X (K_X / L_X)^{\alpha_2} L_X^{\alpha_1+\alpha_2-1}$

(5) $K = K_X + K_Y$

(6) $\left(\frac{w}{\alpha_1}\right)^{1-\alpha_2}\left(\frac{r}{\alpha_2}\right)^{\alpha_2} L_X^{(1-\alpha_1-\alpha_2)} = A_X K^{\alpha_3}$

(7) $\left(\frac{w}{\beta_1}\right)^{1-\beta_2}\left(\frac{r}{\beta_2}\right)^{\beta_2} L_Y^{1-\beta_1-\beta_2} = A_Y K^{\beta_3} p$

(8) $L = L_X + L_Y$

(9) $\frac{bC_Y}{\bar{b}C_X} = \frac{1}{p^{1+\sigma}}$

(10) $U = \left[b^{1/(1+\sigma)} C_X^{\sigma/(1+\sigma)} + \bar{b}^{1/(1+\sigma)} C_Y^{\sigma/(1+\sigma)}\right]^{(1+1/\sigma)(1-\gamma)} /(1-\gamma)$

(11) $C_Y = Y + \overline{Q} + \beta T$

(12) $C_X = X + p*T - p*(\overline{Q} + \beta T)$

(13) $1 + r = \frac{1}{1-\rho}$

(14) $pr = p - p*$

(15) $EX = X - C_X - (K - K_{-1})$

There are 15 equations and 15 variables. Equation (13) implies $r = \frac{1}{1-\rho} - 1$.

Calibration of the reference steady state

We choose the following values *a priori*

$p = 1$, $X = 0.5, Y = 1$, $L = 10$,

$\alpha_1 = 0.60$, $\alpha_2 = 0.10$, $\beta_1 = 0.40$, $\beta_2 = 0.40$, $\beta_3 = 0.50$

$T = 0.01$, $\beta = 0$

$\rho = 0.05$, $\sigma = -0.5$, $\gamma = 0.5$

We assume that the quota not connected to foreign aid, represents 20% of domestic output in importable good, and that the foreign price of importable good is equal to 90% of its domestic price.

We compute

$\overline{Q} = 0.2Y$

$p* = 0.9p$

$r = \frac{1}{1-\rho} - 1$

$K_Y = \beta_2 pY / r$

$K_X = \alpha_2 X / r$

(5) $K = K_X + K_Y$

$w = (\alpha_1 X + \beta_1 pY) / L$

$L_Y = \beta_1 pY / w$

(8) $L_X = L - L_Y$

(1) $A_X = X / (L_X^{\alpha_1} K_X^{\alpha_2})$

(2) $A_Y = Y / (K^{\beta_3} L_Y{}^{\beta_1} K_Y{}^{\beta_2})$

(11) $C_Y = Y + \overline{Q} + \beta T$

(12) $C_X = X + p^* T - p^* (\overline{Q} + \beta T)$

(9) $b = \frac{C_X}{C_X + C_Y p^{1+\sigma}}$

$\overline{b} = 1 - b$

(10) $U = \left[b^{1/(1+\sigma)} C_X^{\sigma/(1+\sigma)} + \overline{b}^{1/(1+\sigma)} C_Y^{\sigma/(1+\sigma)} \right]^{(1+1/\sigma)(1-\lambda)} / (1-\lambda)$

(14) $m = [p^* + \beta(p - p^*)] \frac{C_Y}{C_X + pC_Y} = [p^* + \beta(p - p^*)] \frac{\overline{b}}{\overline{b}p + bp^{1+\sigma}}$

If we use the command steady of DYNARE we get

STEADY-STATE RESULTS:

CX	0.329
CY	1.2
EX	0.171
K	12.35
KX	0.95
KY	11.4
LX	7.5
LY	2.5
p	1

pr	0.1
r	0.0526316
U	2.47305
w	0.04
X	0.5
Y	1

Simulation

We permanently increase foreign aid from T=0.01 to 0.02.

The article includes the results of much more interesting simulations.